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Some inequalities for sum and product of positive semidefinite matrices

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Abstract

The purpose of this paper is to present some inequalities on majorization, unitarily invariant norm, trace, and eigenvalue for sum and product of positive semidefinite (Hermitian) matrices. Some open questions proposed by Marshall and Olkin are resolved. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let A be an $n \times n$ complex matrix. Denote the eigenvalues of A by $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ and singular values of A by $\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)$, and let

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$$\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)), \qquad \sigma(A) = (\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A)).$$

We further assume that the eigenvalues, if they are all real, and the singular values are arranged in decreasing order. As usual, we write $A \geqslant 0$ if A is positive semidefinite (nonnegative definite), $A > 0$ if $A \geqslant 0$ and A is nonsingular, and $A \geqslant B$ if $A - B \geqslant 0$ for Hermitian matrices A and B. An identity matrix is denoted by I. Throughout the paper we assume that all the matrices are $n \times n$ unless otherwise stated.

We first revisit a Fan–Hoffman inequality [3] or [4, p. 266]: If $A \geqslant 0$, then for all unitary matrices U

$$\sigma(A - I) \prec_w \sigma(A - U) \prec_w \sigma(A + I).$$

Here $\prec_w$ stands for weak majorization, that is, $x \prec_w y$ means that every partial sum of the real vector x is dominated by the corresponding partial sum of the vector y, where x and y are real vectors with components arranged in decreasing order. Besides, we write $x \leqslant y$ if x is dominated by y entrywise.

We demonstrate that a more general version

$$\sigma(A - B) \prec_w \sigma(A - BU) \prec_w \sigma(A + B),$$

where $A, B \geqslant 0$ and U is unitary, does not hold in general. But, with the middle term removed, it is true that for all $A, B \geqslant 0$

$$\sigma(A - B) \prec_w \sigma(A + B).$$

This will follow from a stronger log-majorization inequality (Theorem 1).

We then turn our attention to answering some questions raised by Marshall and Olkin, generalizing the results on Euclidean norm to unitarily invariant norm.

After this, in Section 4, we show the trace inequality that for any positive semidefinite matrices A, B and contraction matrices U, V

$$\operatorname{tr}(A - B) \leqslant \operatorname{tr}|A - UBV| \leqslant \operatorname{tr}(A + B),$$

where $|X| = (X^*X)^{1/2}$ (Theorem 3).

In Section 5, we examine the eigenvalues of matrix product. Recall that if $A \geqslant 0$ then $\lambda(A) \geqslant \lambda(A_{11} \oplus 0)$, where A_{11} is any principal submatrix of A. This does not generalize to the product AB, where $A, B \geqslant 0$, though, as is well known, the eigenvalues of AB are nonnegative (AB is not Hermitian in general). We have (Theorem 4), however, for any $A > 0$ and $B \geqslant 0$,

$$\lambda(A^{-1}B) \geqslant \lambda(A_{11}^{-1}B_{11} \oplus 0).$$

In addition, we show (Theorem 6) that if $A \geqslant 0, B \geqslant C \geqslant 0$, then

$$\lambda((A + B)^+B) \geqslant \lambda((A + C)^+C).$$

2. Majorization inequality

We adopt the notation $\log x \prec_w \log y$ to mean that every partial product of x is less than or equal to the corresponding partial product of y, where x and y are vectors with nonnegative components in decreasing order, and use $\|\cdot\|_{\mathrm{ui}}$ for any unitarily invariant norm on the matrix space.

Theorem 1. *Let A and B be positive semidefinite matrices of the same size. Then*

$$\log\sigma(A-B) \prec_w \log\sigma(A+B). \tag{1}$$

As a consequence

$$\sigma(A-B) \prec_w \sigma(A+B). \tag{2}$$

Thus

$$\|A-B\|_{\mathrm{ui}} \leqslant \|A+B\|_{\mathrm{ui}}. \tag{3}$$

Proof. First notice that for any positive semidefinite A and B of the same size

$$|\det(A-B)| \leqslant \det(A+B)$$

by simultaneous congruence of A and B to diagonal matrices. Rewrite this inequality as

$$\prod_{i=1}^{n}\sigma_i(A-B) \leqslant \prod_{i=1}^{n}\sigma_i(A+B).$$

Since $A-B$ is Hermitian, there exists a unitary matrix U such that

$$U(A-B)U^* = \mathrm{Diag}(d_1, d_2, \ldots, d_n)$$

with

$$|d_i| = \sigma_i(A-B), \quad i = 1, 2, \ldots, n.$$

For each positive integer $k, 1 \leqslant k \leqslant n$, let U_1 be the submatrix consisting of the first k rows of U, and U_2 be the rest rows of U. Then by the above argument and the eigenvalue interlacing theorem

$$\begin{aligned}\prod_{i=1}^{k}\sigma_i(A-B) &= \prod_{i=1}^{k}\sigma_i(U_1(A-B)U_1^*)\\ &\leqslant \prod_{i=1}^{k}\sigma_i(U_1(A+B)U_1^*) \leqslant \prod_{i=1}^{k}\sigma_i(A+B).\end{aligned}$$

That is, $\log\sigma(A-B) \prec_w \log\sigma(A+B)$. Inequality (2) then follows since the log-majorization implies weak majorization [4, p. 117]. The norm inequality is immediate due to the fact that $\sigma(A) \prec_w \sigma(B) \Longleftrightarrow \|A\|_{\mathrm{ui}} \leqslant \|B\|_{\mathrm{ui}}$ [4, p. 264]. $\square$

We note that $\sigma(A-B) \prec_w \sigma(A-BU) \prec_w \sigma(A+B)$ does not hold in general for $A, B \geqslant 0$ and unitary U. Take, for example,

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\sigma_1(A-B) = 4 > \sigma_1(A-BU) = 3.6226$.

3. Questions of Marshall and Olkin

This section aims to resolve the problems proposed by Marshall and Olkin in their book [4, ch. 10, Section B, pp. 269–270].

Denote $A_\lambda = \mathrm{Diag}(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$ and $A_\sigma = \mathrm{Diag}(\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))$. The questions (below labeled as in the book) asked by Marshall and Olkin are whether the following results also hold for unitarily invariant norms:

B.8. Let A and B be the complex matrices, and U and V be unitary matrices satisfying the singular value decomposition $B^*A = U(B^*A)_\sigma V^*$. Then for all unitary matrices Γ

$$\|A - B(UV^*)\|_E \leqslant \|A - B\Gamma\|_E. \tag{4}$$

Note that matrices A and B need not be real, and that there is a misprint in the book (second print): $\geqslant$ in the inequality of B.8 should be $\leqslant$. The proof of this inequality is a straightforward computation by writing the Euclidean norm as the square root of trace.

B.8.a. Let A and B be complex matrices and let $A = U_1 A_\sigma V_1$ and $B = U_2 B_\sigma V_2$, where U_1, U_2 and V_1, V_2 are unitary matrices. Then for any unitary matrices U and V

$$\|A - U_1 U_2^* B V_2^* V_1\|_E \leqslant \|A - UBV\|_E. \tag{5}$$

Note that there are also misprints for this inequality in the book: V_1^* should be U_1 and U_1^* be V_1. (The misprints in the two places have not been taken into account in [1,2].)

B.9. Let A and B be complex normal matrices. Then for some permutation matrix P

$$\min_{VV^*=I} \|A_\lambda - VB_\lambda V^*\|_E = \|A_\lambda - PB_\lambda P'\|_E. \tag{6}$$

We now discuss whether these inequalities hold for unitarily invariant norms.

For B.8, the answer is negative. Let, for example,

$$A = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $UV^* = I_2$ and $\sigma(A - B(UV^*)) = (6,6) \not\prec_w \sigma(A - B\Gamma) = (10,0)$.

In fact for any fixed unitary matrix U_0, the general inequality

$$\|A - BU_0\|_{\mathrm{ui}} \leqslant \|A - B\Gamma\|_{\mathrm{ui}}$$

cannot hold for all unitary matrices Γ. Suppose otherwise. We must have, on one hand,

$$\sigma_1(A - BU_0) \leqslant 6, \qquad \sigma_1(A - BU_0) + \sigma_2(A - BU_0) \leqslant 10.$$

On the other hand, let

$$H = (A - BU_0)^*(A - BU_0) = A^2 + U_0^* B^2 U_0 - 7(U_0^* + U_0) =: R + S - T.$$

Then

$$\sigma_1^2(A - BU_0) = \lambda_1(H) \geqslant \lambda_1(R) + \lambda_n(S) - \lambda_1(T) \geqslant 49 + 1 - 14 = 36.$$

It follows that $\sigma_1(A - BU_0) = 6$. Notice also that

$$\begin{aligned} \sigma_1^2(A - BU_0) + \sigma_2^2(A - BU_0) &= \mathrm{tr}(H) = \mathrm{tr}(R) + \mathrm{tr}(S) - \mathrm{tr}(T) \\ &= 100 - \mathrm{tr}(T) \geqslant 72. \end{aligned}$$

Thus $\sigma_2^2(A - BU_0) \geqslant 36$, that is, $\sigma_1(A - BU) + \sigma_2(A - BU) \geqslant 12$, a contradiction.

Similarly, by noting that $\sigma(A - B(-I)) = (8,8)$, we can prove that in general there is no unitary matrix V_0 such that for all unitary matrices Γ,

$$\|A - BV_0\|_{\mathrm{ui}} \geqslant \|A - B\Gamma\|_{\mathrm{ui}}.$$

For B.8.a, the answer is affirmative. Recall (cf. [5] or [7, p. 113]) that for any complex matrices A and B

$$|\sigma(A) - \sigma(B)| \prec_w \sigma(A - B), \tag{7}$$

where $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$. It follows that for any complex matrices A, B and unitary matrices U, V

$$\begin{aligned} \sigma(A_\sigma - B_\sigma) &= |\sigma(A) - \sigma(B)| \prec_w \sigma(A - UBV) \prec_w \sigma(A) + \sigma(B) \\ &= \sigma(A_\sigma + B_\sigma). \end{aligned}$$

Thus we have the following result.

Theorem 2. *For any complex matrices A, B, unitary matrices U, V, and unitarily invariant norm $\|\cdot\|_{\mathrm{ui}}$,*

$$\|A_\sigma - B_\sigma\|_{\mathrm{ui}} \leqslant \|A - UBV\|_{\mathrm{ui}} \leqslant \|A_\sigma + B_\sigma\|_{\mathrm{ui}}. \tag{8}$$

The inequality in B.8.a for unitarily invariant norms follows at once, since

$$\|A - U_1U_2^*BV_2^*V_1\|_{\mathrm{ui}} = \|U_1^*AV_1^* - U_2^*BV_2^*\|_{\mathrm{ui}} = \|A_\sigma - B_\sigma\|_{\mathrm{ui}}.$$

The inequalities in (8) may be rewritten in two-sided form as follows. For any unitary matrices U, V

$$\|A - U_0BV_0\|_{\mathrm{ui}} \leqslant \|A - UBV\|_{\mathrm{ui}} \leqslant \|A + U_0BV_0\|_{\mathrm{ui}},$$

where $U_0 = U_1U_2^*,\ V_0 = V_2^*V_1$. Likewise, for (4), one has for all unitary matrices Γ

$$\|A - BW_0\|_E \leqslant \|A - B\Gamma\|_{\overline{E}} \leqslant \|A + BW_0\|_E,$$

where $W_0 = UV^*$ and U, V are the unitarily matrices in the polar decomposition $B^*A = U(B^*A)_\sigma V^*$.

The answer to B.9 for unitary invariant norm is negative. The question is equivalent to whether the inequality, given normal matrices A and B,

$$\|A_\lambda - PB_\lambda P'\|_{\mathrm{ui}} \leqslant \|A_\lambda - VB_\lambda V^*\|_{\mathrm{ui}} \tag{9}$$

holds for some permutation matrix P and all unitary matrices V.

For a counterexample, let

$$A = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Then

$$VBV^* = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$\sigma(A_\lambda - PB_\lambda P') = (3.1623, 1.4142) \not\prec_w \sigma(A_\lambda - VB_\lambda V^*) = (3.2361, 1.2361).$$

Inequality (9), however, holds for Hermitian matrices. This is seen as follows: If A and B are Hermitian matrices, then (see, e.g., [7, pp. 111, 50])

$$\lambda(A) - \lambda(B) \prec \lambda(A - B) \quad \text{and} \quad |\lambda(A) - \lambda(B)| \prec_w |\lambda(A - B)|.$$

Using this, we have

$$\begin{aligned} \sigma(A_\lambda - B_\lambda) &= |\lambda(A) - \lambda(B)| = |\lambda(A_\lambda) - \lambda(VB_\lambda V^*)| \prec_w |\lambda(A_\lambda - VB_\lambda V^*)| \\ &= \sigma(A_\lambda - VB_\lambda V^*), \end{aligned}$$

which implies (9) with $P = I$.

What is more, one may prove the following identities: Let A and B be normal matrices. Then

$$\min_{VV^*=I} \|A - VBV^*\|_E = \|A_\lambda - PB_\lambda P'\|_E \quad \text{for some permutation matrix } P,$$

$$\max_{VV^*=I} \|A - VBV^*\|_E = \|A_\lambda - PB_\lambda P'\|_E \quad \text{for some permutation matrix } P,$$

$$\min_{VV^*=I} \|A + VBV^*\|_E = \|A_\lambda + PB_\lambda P'\|_E \quad \text{for some permutation matrix } P,$$

$$\max_{VV^*=I} \|A + VBV^*\|_E = \|A_\lambda + PB_\lambda P'\|_E \quad \text{for some permutation matrix } P.$$

Note that none of the above identities holds in general if $\|\cdot\|_E$ is replaced by $\|\cdot\|_{\mathrm{ui}}$.

For Hermitian matrices A and B, we have, by writing $B_\lambda\uparrow = \mathrm{Diag}(\lambda_n(B), \ldots, \lambda_1(B))$,

$$\min_{VV^*=I} \|A - VBV^*\|_{\mathrm{ui}} = \|A_\lambda - B_\lambda\|_{\mathrm{ui}},$$

$$\max_{VV^*=I} \|A - VBV^*\|_{\mathrm{ui}} = \|A_\lambda - B_\lambda\uparrow\|_{\mathrm{ui}},$$

$$\min_{VV^*=I} \|A + VBV^*\|_{\mathrm{ui}} = \|A_\lambda + B_\lambda\uparrow\|_{\mathrm{ui}},$$

$$\max_{VV^*=I} \|A + VBV^*\|_{\mathrm{ui}} = \|A_\lambda + B_\lambda\|_{\mathrm{ui}}.$$

4. Trace inequality

For any complex matrix X, we denote $|X| = (X^*X)^{1/2}$. Recall that X is a contraction matrix if $\sigma_1(X) \leqslant 1$ [8, p. 145] or [9, p. 154]. Note that unitary matrices are contractions.

Theorem 3. *Let A and B be positive semidefinite matrices. Then for any contraction matrices U and V*

$$\mathrm{tr}(A - B) \leqslant \mathrm{tr}\,|A - UBV| \leqslant \mathrm{tr}(A + B). \tag{10}$$

Proof. We first show that if $A \geqslant 0$ then for any contraction matrices U and V

$$\mathrm{Re}\ \mathrm{tr}(A - UAV) \geqslant 0. \tag{11}$$

To see this, let P be a unitary matrix such that $A = PA_\sigma P^*$. Let $P^*VUP = R = (r_{ij})$. Then $|r_{ij}| \leqslant 1$ and

$$\begin{aligned} \text{Re tr}(A - UAV) &= \text{Re tr}(A - AVU) \\ &= \text{Re tr}(A_\sigma - A_\sigma P^* VUP) \\ &= \sum_{i=1}^{n} \sigma_i(A)(1 - \text{Re } r_{ii}) \geqslant 0. \end{aligned}$$

Note that inequality (11) still holds when the negative sign $-$ is replaced by the positive sign $+$.

Now let Q be a unitary matrix such that $A - UBV = |A - UBV|Q^*$ (the polar decomposition). We have

$$\begin{aligned} \text{tr}(A - B) &= \text{Re tr}(A - B) \\ &= \text{Re tr}(A - UBV) - \text{Re tr}(B - UBV) \\ &\leqslant \text{Re tr}(A - UBV) \\ &\qquad \text{(for the second term is nonnegative by (11))} \\ &\leqslant \text{tr}|A - UBV| \\ &= \text{Re tr}(AQ - UBVQ) \\ &= \text{tr}(A + B) - \text{Re tr}(A - AQ) - \text{Re tr}(B + UBVQ) \\ &\leqslant \text{tr}(A + B). \qquad \square \end{aligned}$$

The inequality in the theorem may be rewritten as

$$\sum_{i=1}^{n} \lambda_i(A - B) \leqslant \sum_{i=1}^{n} \sigma_i(A - UBV) \leqslant \sum_{i=1}^{n} \lambda_i(A + B).$$

5. Eigenvalue inequalities

Let A_{11} and B_{11} be corresponding principal submatrices of positive semidefinite matrices A and B, respectively. As is well known, the eigenvalues of AB and $A_{11}B_{11}$ are all nonnegative. The eigenvalue interlacing theorem ensures that $\lambda(A) \geqslant \lambda(A_{11} \oplus 0)$ and $\lambda(B) \geqslant \lambda(B_{11} \oplus 0)$, where 0 is a zero matrix of appropriate size. But the inequality $\lambda(AB) \geqslant \lambda(A_{11}B_{11} \oplus 0)$ does not hold in general: Take, for a counterexample,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

with $A_{11} = 2$ and $B_{11} = 1$. We have, however, the following theorem.

Theorem 4. *Let $A > 0, B \geqslant 0$. Then*

$$\lambda(A^{-1}B) \geqslant \lambda(A_{11}^{-1}B_{11} \oplus 0). \tag{12}$$

Proof. Let A_{11} be of size $k \times k, 1 \leqslant k \leqslant n$, and write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then

$$CAC^* = \begin{pmatrix} A_{11} & 0 \\ 0 & \widetilde{A_{11}} \end{pmatrix}, \qquad C = \begin{pmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{pmatrix}$$

where $\widetilde{A_{11}} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A_{11} in A. Therefore

$$A^{-1} = C^* \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & \widetilde{A_{11}}^{-1} \end{pmatrix} C.$$

Upon computation, we have

$$\begin{aligned}
\lambda(A^{-1}B) &= \lambda\left(\begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & \widetilde{A_{11}}^{-1} \end{pmatrix} CBC^*\right) \\
&= \lambda\left(\begin{pmatrix} A_{11}^{-1/2} & 0 \\ 0 & \widetilde{A_{11}}^{-1/2} \end{pmatrix} \begin{pmatrix} B_{11} & * \\ * & * \end{pmatrix} \begin{pmatrix} A_{11}^{-1/2} & 0 \\ 0 & \widetilde{A_{11}}^{-1/2} \end{pmatrix}\right) \\
&= \lambda\left(\begin{pmatrix} A_{11}^{-1/2}B_{11}A_{11}^{-1/2} & * \\ * & * \end{pmatrix}\right) \\
&\geqslant \lambda(A_{11}^{-1/2}B_{11}A_{11}^{-1/2} \oplus 0) \\
&= \lambda(A_{11}^{-1}B_{11} \oplus 0),
\end{aligned}$$

where $*$'s denote entries irrelevant to our discussions. □

The following theorem is a generalization of a result due to Patel and Toda [6] from trace to eigenvalue. The idea of the proof, given below for completion, is similar to that in [6].

Theorem 5. *Let $A \geqslant 0, B \geqslant C \geqslant 0$, and $A + C > 0$. Then*

$$\lambda((A+B)^{-1}B) \geqslant \lambda((A+C)^{-1}C). \tag{13}$$

Proof. Noticing that $(A+B)^{-1} \leqslant (A+C)^{-1}$, we have

$$\lambda(A^{1/2}(A+B)^{-1}A^{1/2}) \leqslant \lambda(A^{1/2}(A+C)^{-1}A^{1/2})$$

and

$$\begin{aligned}\lambda((A+B)^{-1}B) &= \lambda(I-(A+B)^{-1}A)\\ &= \lambda(I-A^{1/2}(A+B)^{-1}A^{1/2})\\ &\geqslant \lambda(I-A^{1/2}(A+C)^{-1}A^{1/2})\\ &= \lambda(I-(A+C)^{-1}A)\\ &= \lambda((A+C)^{-1}C). \qquad \square\end{aligned}$$

As a corollary [6]

$$\mathrm{tr}((A+B)^{-1}B) \geqslant \mathrm{tr}((A+C)^{-1}C).$$

The above theorem generalizes to Moore–Penrose g-inverses as follows.

Theorem 6. *Let* $A \geqslant 0,\ B \geqslant C \geqslant 0$. *Then*

$$\lambda((A+B)^{+}B) \geqslant \lambda((A+C)^{+}C).$$

Proof. Let rank $(A+B)=r$ and rank $(A+C)=s$. Then there exists a unitary matrix U such that

$$UAU^* = \begin{pmatrix} A_1 & 0\\ 0 & 0\end{pmatrix},\quad UBU^* = \begin{pmatrix} B_1 & 0\\ 0 & 0\end{pmatrix},\quad UCU^* = \begin{pmatrix} C_1 & 0\\ 0 & 0\end{pmatrix},$$

where A_1, B_1, C_1 are of size $r\times r$, $A_1+B_1>0, B_1 \geqslant C_1$. Similarly, for some $r\times r$ unitary matrix V

$$VA_1V^* = \begin{pmatrix} A_2 & 0\\ 0 & 0\end{pmatrix},\quad VB_2V^* = \begin{pmatrix} B_2 & *\\ * & *\end{pmatrix},\quad VC_1V^* = \begin{pmatrix} C_2 & 0\\ 0 & 0\end{pmatrix},$$

where A_2, B_2, C_2 are of size $s\times s$, $A_2+C_2>0, B_2 \geqslant C_2$.

Note that $(M\oplus 0)^{+} = M^{+}\oplus 0$ for any matrix M. We have

$$\begin{aligned}\lambda((A+B)^{+}B) &= \lambda((A_1+B_1)^{-1}B_1\oplus 0_{n-r})\\ &\geqslant \lambda((A_2+B_2)^{-1}B_2\oplus 0_{n-s}) \quad \text{(by Theorem 4)}\\ &\geqslant \lambda((A_2+C_2)^{-1}C_2\oplus 0_{n-s}) \quad \text{(by Theorem 5)}\\ &= \lambda((A_1+C_1)^{+}C_1\oplus 0_{n-r})\\ &= \lambda((A+C)^{+}C). \qquad \square\end{aligned}$$

As a corollary for $A\geqslant 0, B\geqslant C\geqslant 0$,

$$\mathrm{tr}((A+B)^{+}B) \geqslant \mathrm{tr}((A+C)^{+}C).$$

We end the paper by noting that (12) does not generalize to the Moore–Penrose g-inverses. Take, for a counterexample,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $A_{11} = 1$ and $B_{11} = 1$. Then $\lambda(A^+B) = (0.5, 0) \not\succ \lambda(A_{11}^+B_{11} \oplus 0) = (1, 0)$.

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